Chapter 39

9. The discussion on the probability of detection for the one-dimensional case can be readily extended to two dimensions. In analogy to Eq. 39-10, the normalized wave function in two dimensions can be written as

$$\psi_{n_x,n_y}(x,y)=\psi_{n_x}(x)\psi_{n_y}(y)=\left[\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L_x}x\right)\right]\left[\sqrt{\frac{2}{L_y}}\sin\left(\frac{n_y\pi}{L_y}y\right)\right]$$
$$=\sqrt{\frac{4}{L_xL_y}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right).$$

The probability of detection by a probe of dimension $\Delta x\ \Delta y$ placed at (x,y) is

$$p(x,y)=\left|\psi_{n_x,n_y}(x,y)\right|^2\Delta x\ \Delta y=\frac{4(\Delta x\ \Delta y)}{L_xL_y}\sin^2\left(\frac{n_x\pi}{L_x}x\right)\sin^2\left(\frac{n_y\pi}{L_y}y\right).$$

With $L_x=L_y=L=200\text{ pm}$ and $\Delta x=\Delta y=4.00\text{ pm}$, the probability of detecting an electron in $(n_x,n_y)=(1,3)$ state by placing a probe at $(0.200L, 0.800L)$ is

$$p=\frac{4(\Delta x\ \Delta y)}{L_xL_y}\sin^2\left(\frac{n_x\pi}{L_x}x\right)\sin^2\left(\frac{n_y\pi}{L_y}y\right)$$
$$=\frac{4(4.00\text{ pm})^2}{(200\text{ pm})^2}\sin^2\left(\frac{\pi}{L}0.200L\right)\sin^2\left(\frac{3\pi}{L}0.800L\right)$$
$$=4\left(\frac{4.00\text{ pm}}{200\text{ pm}}\right)^2\sin^2\left(0.200\pi\right)\sin^2\left(2.40\pi\right)=5.0\times10^{-4}\ .$$

11. Using $E = hc/\lambda = (1240\,\text{eV}\cdot\text{nm})/\lambda$, the energies associated with λ_a, λ_b and λ_c are

$$E_a = \frac{hc}{\lambda_a} = \frac{1240\,\text{eV}\cdot\text{nm}}{14.588\ \text{nm}} = 85.00\ \text{eV}$$

$$E_b = \frac{hc}{\lambda_b} = \frac{1240\,\text{eV}\cdot\text{nm}}{4.8437\ \text{nm}} = 256.0\ \text{eV}$$

$$E_c = \frac{hc}{\lambda_c} = \frac{1240\,\text{eV}\cdot\text{nm}}{2.9108\ \text{nm}} = 426.0\ \text{eV}.$$

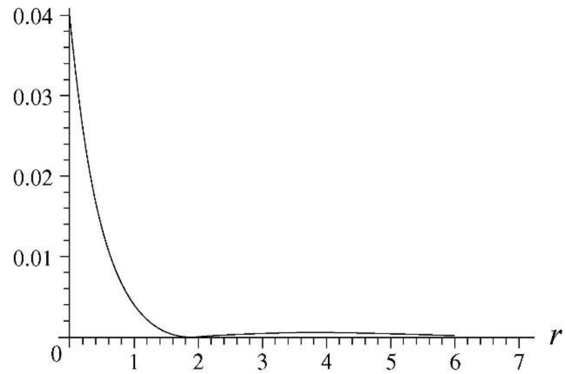
The ground-state energy is

$$E_1 = E_4 - E_c = 450.0\ \text{eV} - 426.0\ \text{eV} = 24.0\ \text{eV}\ .$$

Since $E_a = E_2 - E_1$, the energy of the first excited state is

$$E_2 = E_1 + E_a = 24.0\ \text{eV} + 85.0\ \text{eV} = 109\ \text{eV}\ .$$

15. (a) The plot shown below for $|\psi_{200}(r)|^2$ is to be compared with the dot plot of Fig. 39-21. We note that the horizontal axis of our graph is labeled "r," but it is actually r/a (that is, it is in units of the parameter a). Now, in the plot below there is a high central peak between $r = 0$ and $r \sim 2a$, corresponding to the densely dotted region around the center of the dot plot of Fig. 39-21. Outside this peak is a region of near-zero values centered at $r = 2a$, where $\psi_{200} = 0$. This is represented in the dot plot by the empty ring surrounding the central peak. Further outside is a broader, flatter, low peak that reaches its maximum value at $r = 4a$. This corresponds to the outer ring with near-uniform dot density, which is lower than that of the central peak.



(b) The extrema of $\psi^2(r)$ for $0 < r < \infty$ may be found by squaring the given function, differentiating with respect to r, and setting the result equal to zero:

$$-\frac{1}{32}\frac{(r-2a)(r-4a)}{a^6\pi}e^{-r/a} = 0$$

which has roots at $r = 2a$ and $r = 4a$. We can verify directly from the plot above that $r = 4a$ is indeed a local maximum of $\psi_{200}^2(r)$. As discussed in part (a), the other root ($r = 2a$) is a local minimum.

(c) Using Eq. 39-43 and Eq. 39-41, the radial probability is

$$P_{200}(r) = 4\pi r^2\psi_{200}^2(r) = \frac{r^2}{8a^3}\left(2-\frac{r}{a}\right)^2 e^{-r/a}.$$

(d) Let $x = r/a$. Then

$$\int_0^\infty P_{200}(r)dr = \int_0^\infty \frac{r^2}{8a^3}\left(2-\frac{r}{a}\right)^2 e^{-r/a}dr = \frac{1}{8}\int_0^\infty x^2(2-x)^2e^{-x}dx = \int_0^\infty (x^4-4x^3+4x^2)e^{-x}dx$$
$$= \frac{1}{8}[4!-4(3!)+4(2!)] = 1$$

where we have used the integral formula $\int_0^\infty x^n e^{-x}dx = n!$.

28. (a) We use Eq. 39-44. At $r = 0$, $P(r) \propto r^2 = 0$.

(b) At $r = 1.5a$, $P(r) = \frac{4}{a^3}(1.5a)^2 e^{-3a/a} = \frac{9e^{-3}}{a} = \frac{9e^{-3}}{5.29\times10^{-2}\text{ nm}} = 8.47\,\text{nm}^{-1}$.

(c) At $r = 3a$, $P(r) = \frac{4}{a^3}(3a)^2 e^{-6a/a} = \frac{36e^{-6}}{a} = \frac{36e^{-6}}{5.29\times10^{-2}\text{ nm}} = 1.69\,\text{nm}^{-1}$.

35. **THINK** The ground state of the hydrogen atom corresponds to $n = 1$, $l = 0$, and $m_l = 0$.

EXPRESS The proposed wave function is

$$\psi = \frac{1}{\sqrt{\pi} a^{3/2}} e^{-r/a}$$

where a is the Bohr radius. Substituting this into the right side of Schrödinger's equation, our goal is to show that the result is zero.

ANALYZE The derivative is

$$\frac{d\psi}{dr} = -\frac{1}{\sqrt{\pi} a^{5/2}} e^{-r/a}$$

so

$$r^2 \frac{d\psi}{dr} = -\frac{r^2}{\sqrt{\pi} a^{5/2}} e^{-r/a}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) = \frac{1}{\sqrt{\pi} a^{5/2}}\left[-\frac{2}{r}+\frac{1}{a}\right]e^{-r/a} = \frac{1}{a}\left[-\frac{2}{r}+\frac{1}{a}\right]\psi.$$

The energy of the ground state is given by $E = -me^4/8\varepsilon_0^2 h^2$ and the Bohr radius is given by $a = h^2\varepsilon_0/\pi m e^2$, so $E = -e^2/8\pi\varepsilon_0 a$. The potential energy is given by

$$U = -e^2/4\pi\varepsilon_0 r\,,$$

so

$$\frac{8\pi^2 m}{h^2}[E-U]\psi = \frac{8\pi^2 m}{h^2}\left[-\frac{e^2}{8\pi\varepsilon_0 a}+\frac{e^2}{4\pi\varepsilon_0 r}\right]\psi = \frac{8\pi^2 m}{h^2}\frac{e^2}{8\pi\varepsilon_0}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi$$
$$= \frac{\pi m e^2}{h^2\varepsilon_0}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi = \frac{1}{a}\left[-\frac{1}{a}+\frac{2}{r}\right]\psi.$$

The two terms in Schrödinger's equation cancel, and the proposed function ψ satisfies that equation.

LEARN The radial probability density of the ground state of hydrogen atom is given by Eq. 39-44:

$$P(r) = |\psi|^2\,(4\pi r^2) = \frac{1}{\pi a^3} e^{-2r/a}(4\pi r^2) = \frac{4}{a^3} r^2 e^{-2r/a}\,.$$

A plot of $P(r)$ is shown in Fig. 39-20.