15. (a) and (b) Letting $a = 5.292 \times 10^{-11}$ m be the Bohr radius, the potential energy becomes

$$U = -\frac{e^2}{4\pi\varepsilon_0 a} = \frac{\left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2\right) \left(1.602 \times 10^{-19} \text{ C}\right)^2}{5.292 \times 10^{-11} \text{ m}} = -4.36 \times 10^{-18} \text{ J} = -27.2 \text{ eV} .$$

The kinetic energy is K = E - U = (-13.6 eV) - (-27.2 eV) = 13.6 eV.

21. Since $E_n \propto L^{-2}$ in Eq. 39-4, we see that if *L* is doubled, then E_1 becomes (2.6 eV)(2)⁻² = 0.65 eV.

23. **THINK** The ground state of the hydrogen atom corresponds to n = 1, $\ell = 0$, and $m_{\ell} = 0$.

EXPRESS The proposed wave function is

$$\psi = \frac{1}{\sqrt{\pi}a^{3/2}} e^{-r/a}$$

where a is the Bohr radius. Substituting this into the right side of Schrödinger's equation, our goal is to show that the result is zero.

ANALYZE The derivative is

$$\frac{d\psi}{dr} = -\frac{1}{\sqrt{\pi}a^{5/2}}e^{-r/a}$$

so

$$r^{2} \frac{d\psi}{dr} = -\frac{r^{2}}{\sqrt{\pi a^{5/2}}} e^{-r/a}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{1}{\sqrt{\pi}a^{5/2}}\left[-\frac{2}{r} + \frac{1}{a}\right]e^{-r/a} = \frac{1}{a}\left[-\frac{2}{r} + \frac{1}{a}\right]\psi.$$

The energy of the ground state is given by $E = -me^4/8\varepsilon_0^2h^2$ and the Bohr radius is given by $a = h^2\varepsilon_0/\pi me^2$, so $E = -e^2/8\pi\varepsilon_0 a$. The potential energy is given by

$$U=-e^2/4\pi\varepsilon_0 r\,,$$

so

$$\frac{8\pi^2 m}{h^2} [E - U] \psi = \frac{8\pi^2 m}{h^2} \left[-\frac{e^2}{8\pi\varepsilon_0 a} + \frac{e^2}{4\pi\varepsilon_0 r} \right] \psi = \frac{8\pi^2 m}{h^2} \frac{e^2}{8\pi\varepsilon_0} \left[-\frac{1}{a} + \frac{2}{r} \right] \psi$$
$$= \frac{\pi m e^2}{h^2 \varepsilon_0} \left[-\frac{1}{a} + \frac{2}{r} \right] \psi = \frac{1}{a} \left[-\frac{1}{a} + \frac{2}{r} \right] \psi.$$

The two terms in Schrödinger's equation cancel, and the proposed function ψ satisfies that equation.

LEARN The radial probability density of the ground state of hydrogen atom is given by Eq. 39-44:

$$P(r) = |\psi|^2 (4\pi r^2) = \frac{1}{\pi a^3} e^{-2r/a} (4\pi r^2) = \frac{4}{a^3} r^2 e^{-2r/a}.$$

A plot of P(r) is shown in Fig. 39-20.

24. We follow Sample Problem — "Detection potential in a 1D infinite potential well" in the presentation of this solution. The integration result quoted below is discussed in a little more detail in that Sample Problem. We note that the arguments of the sine functions used below are in radians.

(a) The probability of detecting the particle in the region $0 \le x \le L/4$ is

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{0}^{\pi/4}\sin^{2}y\,dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)\Big|_{0}^{\pi/4} = 0.091.$$

(b) As expected from symmetry,

$$\left(\frac{2}{L}\right)\left(\frac{L}{\pi}\right)\int_{\pi/4}^{\pi}\sin^2 y\,dy = \frac{2}{\pi}\left(\frac{y}{2} - \frac{\sin 2y}{4}\right)\Big|_{\pi/4}^{\pi} = 0.091.$$

(c) For the region $L/4 \le x \le 3L/4$, we obtain

$$\left(\frac{2}{L}\right) \left(\frac{L}{\pi}\right) \int_{\pi/4}^{3\pi/4} \sin^2 y \, dy = \frac{2}{\pi} \left(\frac{y}{2} - \frac{\sin 2y}{4}\right) \Big|_{\pi/4}^{3\pi/4} = 0.82$$

which we could also have gotten by subtracting the results of part (a) and (b) from 1; that is, 1 - 2(0.091) = 0.82.

49. For an electron inside a cubical box of widths $L_x = L_y = L_z = L$, the quantized energies are given by, in multiple of $h^2/8mL^2$,

$$\frac{E_{n_x,n_y,n_z}}{h^2/8mL^2} = \left(n_x^2 + n_y^2 + n_z^2\right)$$

The frequencies emitted are given by $f = \Delta E/h$. For a transition from the second excited state (2, 2, 1) to the ground state (1, 1, 1), we find

$$f = (9.00 - 3.00) \left(\frac{h}{8mL^2}\right) = (6.00) \left(\frac{h}{8mL^2}\right).$$

In the following, we omit the $h/8mL^2$ factors. For a transition between the fourth excited state and the ground state, we have f = 12.00 - 3.00 = 9.00. For a transition between the third excited state and the ground state, we have f = 11.00 - 3.00 = 8.00. For a transition between the third excited state and the first excited state, we have f = 11.00 - 6.00 = 5.00. For a transition between the fourth excited state and the third excited state, we have f = 12.00 - 11.00 = 1.00. For a transition between the third excited state, we have f = 12.00 - 11.00 = 1.00. For a transition between the third excited state and the second excited state, we have f = 11.00 - 9.00 = 2.00. For a transition between the second excited state and the first excited state, we have f = 3.00, which also results from some other transitions.

- (a) From the above, we see that there are 7 frequencies.
- (b) The lowest frequency is, in units of $h/8mL^2$, 1.00.
- (c) The second lowest frequency is, in units of $h/8mL^2$, 2.00.
- (d) The third lowest frequency is, in units of $h/8mL^2$, 3.00.
- (e) The highest frequency is, in units of $h/8mL^2$, 9.00.
- (f) The second highest frequency is, in units of $h/8mL^2$, 8.00.
- (g) The third highest frequency is, in units of $h/8mL^2$, 6.00.

51. Using $E = hc / \lambda = (1240 \text{ eV} \cdot \text{nm})/\lambda$, the energies associated with λ_a , λ_b and λ_c are

$$E_{a} = \frac{hc}{\lambda_{a}} = \frac{1240 \text{ eV} \cdot \text{nm}}{14.588 \text{ nm}} = 85.00 \text{ eV}$$
$$E_{b} = \frac{hc}{\lambda_{b}} = \frac{1240 \text{ eV} \cdot \text{nm}}{4.8437 \text{ nm}} = 256.0 \text{ eV}$$
$$E_{c} = \frac{hc}{\lambda_{c}} = \frac{1240 \text{ eV} \cdot \text{nm}}{2.9108 \text{ nm}} = 426.0 \text{ eV}.$$

The ground-state energy is

$$E_1 = E_4 - E_c = 450.0 \text{ eV} - 426.0 \text{ eV} = 24.0 \text{ eV}$$
.

Since $E_a = E_2 - E_1$, the energy of the first excited state is

$$E_2 = E_1 + E_a = 24.0 \text{ eV} + 85.0 \text{ eV} = 109 \text{ eV}$$